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ASYMPTOTIC SOLUTION TO THE TANGENTIAL LOW THRUST ENERGY INCREASE TRAJECTORY

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16. ABSTRACT

An approximate analytic solution to the low-thrust constant acceleration energy increase trajectory using a tangential steering program is presented. The solution is based on a two-variable asymptotic expansion of the equations of motion formulated in regularized variables. The high accuracy of the second order solution derived is demonstrated by comparing it with numerically integrated trajectories.

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LIST OF SYMBOLS

Symbol	Definition
F	Thrust
h	Keplerian energy
m	Vehicle mass
r	Radius
s	Regularized time, independent variable
t	Time
u	Velocity component in radius direction
v	Velocity component in circumferential direction
$\bar{v}$	Velocity vector
x	Variable in general
A,B	Functions of τ_ϵ, the slow time variable only, related to the first order solution of r, u
E,F	Functions of τ_ϵ only, related to the second order solution of r, u
H	Function of τ_ϵ related to h
R	Function of τ_ϵ related to r
V	Function of τ_ϵ related to v
C,K	Constants

Greek Symbols

α	Path angle
β	Mass flow rate, $\dot{m}$
ϵ	Thrusting acceleration, $\epsilon = \dfrac{F}{m_i}$

LIST OF SYMBOLS (Concluded)

Symbol	Definition
φ	Central angle
ϕ	Function of τ_ϵ only, related to φ
μ	Gravitational parameter $\mu = GM$ with G the gravitational constant and M the mass of the central body
ω	Function of τ_ϵ in definition equation of τ
τ	Fast time scale
τ_ϵ	Slow time scale

Subscripts

i	Initial
ϵ	Slow time variable in combination with τ_ϵ
1	Partial derivative with respect to τ_ϵ
2	Partial derivative with respect to τ, the fast time scale

Superscripts

(i)	i = 0, 1, 2; the order of the solution
s, t	Left-hand superscript s or t indicates the independent variable to which the dependent variable is related

Other Symbols

$\bar{a}$	Indicates a as vector
$\lvert\bar{a}\rvert$	Absolute value of vector a
$\frac{\partial}{\partial x}$	Partial derivative with respect to x
a'	Prime denotes derivative with respect to s

TECHNICAL MEMORANDUM X-64833

ASYMPTOTIC SOLUTION TO THE TANGENTIAL LOW THRUST ENERGY INCREASE TRAJECTORY

I. INTRODUCTION

A spacecraft, equipped with a low-thrust constant acceleration propulsion device. ascending from an initial circular orbit around a spherical central body describes a spiral orbit. To determine the radius vector and velocity of the spacecraft at any time, the equations of motion need to be integrated numerically, even in the case of a tangential thrust steering program which is considered in this study. It is well-known [1] that for energy increase trajectories, the tangential thrust steering is very close to the optimal steering program, especially in the inner, multirevolution part of the spiral trajectory, which is characterized by the acting of a small force on a space vehicle in the presence of a strong gravitational field. With the absence of any other perturbing force, nearly Keplerian orbit conditions exist locally. The deviations between the optimal and the tangential thrust trajectories arise in the outer part on which the vehicle moves from near circular conditions to parabolic velocity and the gravitational force drops to the same order of magnitude as the thrust.

To gain some experience in performing the analysis of a very accurate analytical approximate solution of the optimal trajectory, the tangential thrust trajectory offers a very good simplified problem, since it possesses all the main characteristics of the optimal solution.

In the literature, a number of analytical approximations of the optimal trajectory are presented, and many are available that consider the tangential thrust trajectory [2,3]. The method used herein to derive a solution for the tangential thrust trajectory follows those of Reference 2. The basic method is described by Cole and Kevorkian [4] based on previous work of Linstedt and Poincaré [5,6]. However, formulation of the equation of motion using regularized variables, as presented in Reference 1, allows one to derive an extremely accurate second order solution using a very simple analysis.

As will be shown in Section IV, in comparison to numerically calculated tangential thrust trajectories, the analytical solution derived is extremely accurate. The accuracy decreases with time, which indicates that in addition to the basic characteristic limitations previously discussed, accuracy requirements will determine the limits of the applicability of the solution presented.

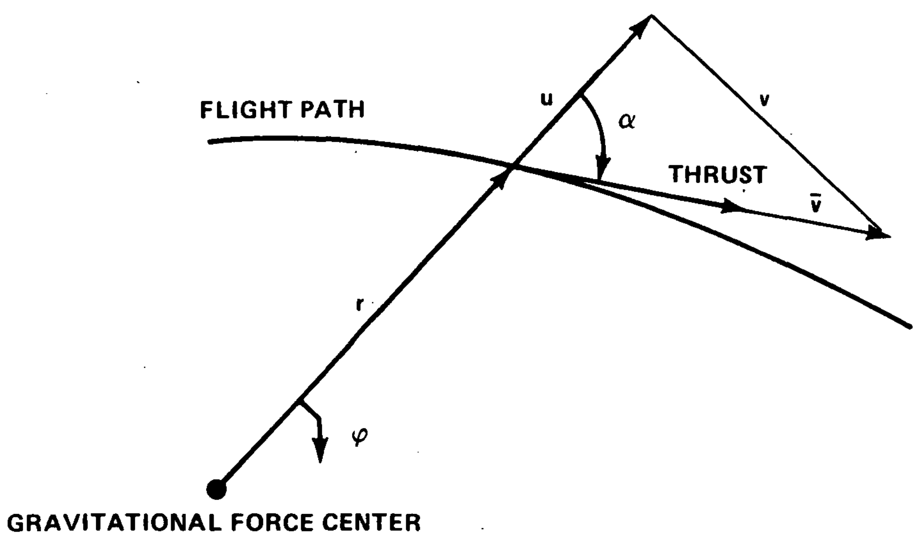
For $\epsilon \leqslant 0.001$, the solution for all of the state variables gives at least three-digit accuracy, up to an energy level close to escape. The solution fails to describe the trajectory near escape.

Sections II and III contain the derivation of the solution which is summarized in Section III.E. Section IV shows an alternate time approximation. The solution is discussed in Section V.

II. MATHEMATICAL FORMULATION

The ascent of a spacecraft is considered from an initial circular orbit around a spherical planet. The vehicle is described by a point mass and is equipped with a low-thrust constant acceleration engine. The engine produces a small continuous thrusting force. No other perturbations are present. The trajectory is considered as planar.

The following figure shows the definitions of symbols used for the mathematical description of the spacecraft motion.



The equations of motion are formulated as follows:

$$r' = u$$

$$\varphi' = v/r \qquad (1)$$

$$u' = \mu + 2hr + \epsilon r^2 \frac{u}{|\bar{v}|}$$

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$$v' = \epsilon r^2 \frac{v}{|\bar{v}|}$$

$$h' = \epsilon |\bar{v}| \qquad \text{(1) (Concluded)}$$

$$t' = r$$

with the absolute value of the vehicle velocity

$$|\bar{v}| = \sqrt{u^2 + v^2} \quad , \qquad (2)$$

and the thrust to mass ratio

$$\epsilon = F/m \quad . \qquad (3)$$

The acceleration is considered as constant,

$$\epsilon = \text{constant} \quad . \qquad (4)$$

Using the initial orbit as a scaling system, the initial conditions are

$$\begin{aligned} r(s = 0) &= r(0) = 1 \\ \varphi(s = 0) &= \varphi(0) = 0 \\ u(s = 0) &= u(0) = 0 \\ v(s = 0) &= v(0) = 1 \\ h(s = 0) &= h(0) = -0.5 \\ t(s = 0) &= t(0) = 0 \end{aligned} \qquad (5)$$

Equation system (1) together with definition equations (2) and (3) and the boundary conditions (5) describe the tangential thrust spiral orbit of a low-thrust space vehicle. In this particular formulation, the energy h is used as an additional (or redundant) variable.

The independent variable s is defined by the last equation of system (1). (For detailed information, see Reference 1.)

Although the system (1) cannot be solved analytically, the solution can be approximated by use of a two-variable expansion procedure. The variables will be evaluated by an asymptotic expansion in powers of the small parameter ϵ,

$$x = x^{(0)} + \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \epsilon^3 x^{(3)} + 0(\epsilon^4) \quad , \tag{6}$$

with x in equation (6) indicating any variable.

In addition, the variables are considered as functions of two new independent variables defined by

$$\begin{aligned} \tau_\epsilon &= \epsilon s \\ \tau &= \int_0^s \omega(\tau_\epsilon) d\sigma \end{aligned} \tag{7}$$

The derivatives of equations (7) with respect to the independent variables are

$$\begin{aligned} \tau_\epsilon' &= \epsilon \\ \tau_\epsilon^{(n)} &= 0 \\ \tau' &= \omega(\tau_\epsilon) \\ \tau^{(n)} &= \epsilon^n \frac{\partial^{n-1}\omega}{\partial \tau_\epsilon^{n-1}} \end{aligned} \quad , \tag{8}$$

with $n \geqslant 2$. The derivations of a variable with respect to the new independent variables follow therefrom:

$$
\begin{aligned}
x' &= x_1 \tau_\epsilon' + x_2 \tau' = \epsilon x_1 + \omega(\tau_\epsilon) x_2 \\
x'' &= x_{11} (\tau_\epsilon')^2 + x_{22} (\tau')^2 + x_2 \tau'' = \epsilon^2 x_{11} + \omega^2 x_{22} + \epsilon^2 \frac{\partial \omega}{\partial \tau_\epsilon} x_2
\end{aligned}
\tag{9}
$$

The subscripts 1 and 2 denote the partial derivatives defined as follows:

$$
\begin{aligned}
x_1 &= \frac{\partial x}{\partial \tau_\epsilon} \\
x_2 &= \frac{\partial x}{\partial \tau} \\
x_{11} &= \frac{\partial^2 x}{\partial \tau_\epsilon^2} \\
x_{22} &= \frac{\partial^2 x}{\partial \tau^2} \\
x_{12} &= \frac{\partial^2 x}{\partial \tau_\epsilon \partial \tau}
\end{aligned}
\tag{10}
$$

To derive an asymptotic solution of equations (1), the independent variable of the system is transformed to the two independent variables defined by equations (7). After expanding the right side of equations (1) using power series of the variables as given in equation (6), collecting terms of the same power of ϵ will lead to sets of partial differential equations whose solution is given in Section III.

III. ASYMPTOTIC EXPANSION SOLUTION

A. Base or Zero Order Solution

Performing as stated in Section II will lead to the following zero order differential equations:

$$
\left.
\begin{aligned}
\omega r_2^{(0)} &= u^{(0)} \\
\omega \varphi_2^{(0)} &= v^{(0)}/r^{(0)} \\
\omega u_2^{(0)} &= 1 + 2h^{(0)}\, r^{(0)} \\
\omega v_2^{(0)} &= 0 \\
\omega h_2^{(0)} &= 0 \\
\omega t_2^{(0)} &= r^{(0)}
\end{aligned}
\right\} , \qquad (11)
$$

with boundary conditions

$$
\begin{aligned}
r^{(0)}(0,0) &= 1 \\
\varphi^{(0)}(0,0) &= 0 \\
u^{(0)}(0,0) &= 0 \\
v^{(0)}(0,0) &= 1 \\
h^{(0)}(0,0) &= -0.5 \\
t^{(0)}(0,0) &= 0
\end{aligned}
\qquad (12)
$$

Instead of using the complete solution of systems (11) and (12) describing a generalized Keplerian solution of the equations of motion without thrust, a partial solution is used as a base solution as follows:

$$
\begin{aligned}
r^{(0)} &= R^{(0)}(\tau_\epsilon) \\
\varphi^{(0)} &= \omega^{-1}\, V^{(0)}\, R^{(0)^{-1}} \tau + \phi^{(0)}(\tau_\epsilon) \qquad (13) \\
u^{(0)} &= 0
\end{aligned}
$$

$$v^{(0)} = V^{(0)}(\tau_\epsilon)$$

$$h^{(0)} = H^{(0)}(\tau_\epsilon) \qquad . \tag{13 (Concluded)}$$

$$t^{(0)} = R^{(0)}\,\omega^{-1}\,\tau + T^{(0)}(\tau_\epsilon)$$

Inserting this solution into equations (11) will give

$$H^{(0)} = -[2R^{(0)}]^{-1} \qquad . \tag{14}$$

The capital letters denote unknown functions of the slow variable τ_ϵ only; i.e., constants with respect to an integration over the fast variable τ. They are determinable by removing secular τ-terms due to the asymptotic expansion from the first order solution and by satisfying the boundary conditions (12).

B. First Order Solution

The first order differential equations derived are

$$\begin{aligned}
\omega r_2{}^{(1)} &= -\ R_1{}^{(0)} + u^{(1)} \\
\omega \varphi_2{}^{(1)} &= -\ \varphi_1{}^{(0)} + v^{(1)}/R^{(0)} - V^{(0)} r^{(1)}/R^{(0)\,2} \\
\omega u_2{}^{(1)} &= 2\,[R^{(0)}\,h^{(1)} + H^{(0)}\,r^{(1)}] \\
\omega v_2{}^{(1)} &= -\ V_1{}^{(0)} + R^{(0)\,2} \\
\omega h_2{}^{(1)} &= -\ H_1{}^{(0)} + V^{(0)} \\
\omega t_2{}^{(1)} &= -\ t_1{}^{(0)} + r^{(1)}
\end{aligned} \tag{15}$$

with boundary conditions

$$\begin{aligned}
r^{(1)}(0,0) &= 0 \\
\varphi^{(1)}(0,0) &= 0
\end{aligned} \tag{16}$$

$$u^{(1)}(0,0) = 0$$

$$v^{(1)}(0,0) = 0$$

$$h^{(1)}(0,0) = 0 \qquad (16) \text{ (Concluded)}$$

$$t^{(1)}(0,0) = 0$$

Because the right side of the velocity component v and energy equations are independent of τ, it follows that

$$v^{(1)} = [- V_1^{(0)} + R^{(0)^2}] \tau + V^{(1)}(\tau_\epsilon) \qquad (17)$$

and

$$h^{(1)} = [- H_1^{(0)} + V^{(0)}] \tau + H^{(1)}(\tau_\epsilon) \quad . \qquad (18)$$

To determine $H^{(0)}$, $R^{(0)}$, and $V^{(0)}$, the secular terms in equations (17) and (18) will be removed:

$$- V_1^{(0)} + R^{(0)^2} = 0 \qquad (19)$$

and

$$- H_1^{(0)} + V^{(0)} = 0 \quad . \qquad (20)$$

Using equation (14) we find from equations (19) and (20) that

$$V^{(0)^2} = R^{(0)} \qquad (21)$$

and

$$R_1^{(0)} = 2\, R^{(0)\,5/2} \quad , \tag{22}$$

and integration of equation (22) gives

$$R^{(0)} = (1 - 3\tau_\epsilon)^{-2/3} \quad . \tag{23}$$

The derived functions for $H^{(0)}$, $R^{(0)}$, and $V^{(0)}$ fulfill the boundary conditions (12).

With the convenient choice of

$$\omega(\tau_\epsilon) = R^{(0)\,-1/2} \quad , \tag{24}$$

the solution of the remaining equations (15) follows:

$$r^{(1)} = A \sin\tau + B\cos\tau + 2\, R^{(0)^2} H^{(1)} \quad , \tag{25}$$

$$u^{(1)} = R^{(0)^{-1/2}} (A\cos\tau - B\sin\tau) + 2\, R^{(0)^{5/2}} \quad , \tag{26}$$

$$t^{(1)} = -\, 3\, R^{(0)^{7/2}} \tau^2/2 - R^{(0)^{1/2}} (A\cos\tau - B\sin\tau) + T^{(1)}(\tau_\epsilon) \quad , \tag{27}$$

and

$$\varphi^{(1)} = -\, R^{(0)^{-1}} (-\,A\cos\tau + B\sin\tau) + \phi^{(1)}(\tau_\epsilon) \quad . \tag{28}$$

Removing secular τ-terms from equations (27) and (28) gives

$$\phi_1^{(0)} = V^{(1)}/R^{(0)} - 2\, R^{(0)^{1/2}} H^{(1)} \tag{29}$$

and

$$T_1^{(0)} = -\, 2\, R^{(0)^2} H^{(1)} \tag{30}$$

Consequently, a complete base solution is dependent on $H^{(1)}$, which (as well as A, B, $T^{(1)}$, $\phi^{(1)}$, and $V^{(1)}$) may be determined from the second order solution.

The two independent variables, using equation (24), are now

$$\tau_\epsilon = \epsilon s \tag{31}$$

and

$$\tau = \int_0^s \omega(t_\epsilon)\, d\sigma = \int_0^s R^{(0)^{-1/2}}\, d\sigma = \frac{1}{4\epsilon}\left[1 - R^{(0)^{-2}}\right] \quad . \tag{32}$$

C. Second Order Solution

The second order differential equations are

$$\begin{aligned}
\omega\, r_2^{(2)} &= -\, r_1^{(1)} + u^{(2)} \\
\omega\, \varphi_2^{(2)} &= -\, \varphi_1^{(1)} + R^{(0)^{-1}} V^{(2)} - R^{(0)^{-3/2}} r^{(2)} - R^{(0)^{-2}} r^{(1)} v^{(1)} + R^{(0)^{-5/2}} r^{(1)^2} \\
\omega\, u_2^{(2)} &= -\, u_1^{(1)} + 2H^{(0)} r^{(2)} + 2\, R^{(0)} h^{(2)} + 2\, r^{(1)} h^{(1)} + R^{(0)^{3/2}} u^{(1)} \\
\omega\, v_2^{(2)} &= -\, V_1^{(1)} + 2\, R^{(0)} r^{(1)} \\
\omega\, h_2^{(2)} &= -\, h_1^{(1)} + v^{(1)} \\
\omega\, t_2^{(2)} &= -\, t_1^{(1)} + r^{(2)}
\end{aligned} \tag{33}$$

with boundary conditions

$$\begin{aligned}
r^{(2)}(0,0) &= 0 \\
\varphi^{(2)}(0,0) &= 0 \\
u^{(2)}(0,0) &= 0
\end{aligned} \tag{34}$$

$$v^{(2)}(0,0) = 0$$

$$h^{(2)}(0,0) = 0 \qquad (34) \text{ (Concluded)}$$

$$t^{(2)}(0,0) = 0$$

To solve this set of equations we start again with the energy and v-equations

$$v^{(2)} = 2\, R^{(0)^{3/2}} (-A \cos \tau + B \sin \tau) + V^{(2)}(\tau_\epsilon) \qquad (35)$$

and

$$h^{(2)} = H^{(2)}(\tau_\epsilon) \qquad (36)$$

By removing improper secular τ-terms from equations (35) and (36) we get a differential equation for $H^{(1)}$

$$H_{11}{}^{(1)} + 4\, R^{(0)^3} H^{(1)} = 0 \qquad (37)$$

with boundary conditions

$$H^{(1)}(\tau_\epsilon = 0) = V^{(1)}(\tau_\epsilon = 0) = H_1{}^{(1)}(\tau_\epsilon = 0) = 0 \quad .$$

Solving equation (37) and obeying the boundary conditions gives

$$H^{(1)} = V^{(1)} = 0 \qquad (38)$$

This gives

$$T^{(0)} = \phi^{(0)} = 0 \qquad (39)$$

Consequently, a complete zero order solution is known. Equations (15) will be simplified by removing terms which contain $H^{(1)}$ or $V^{(1)}$.

A second order differential equation for the radius is derived from the first equation of system (33) as

$$r_{22}{}^{(2)} = - R^{(0)^{1/2}} r_{12}{}^{(1)} + R^{(0)^{1/2}} u_2{}^{(2)} \tag{40}$$

with

$$r_{12}{}^{(1)} = A_1 \cos\tau - B_1 \sin\tau \tag{41}$$

and

$$u_1{}^{(1)} = R^{(0)^{-1/2}} (A_1 \cos\tau - B_1 \sin\tau) - R^{(0)}(A \cos\tau - B \sin\tau) + 10\, R^{(0)^4} \quad , \tag{42}$$

and the third equation from system (33) with equation (40),

$$\begin{aligned} r_{22}{}^{(2)} + r^{(2)} = &- [- 2\, R^{(0)^{1/2}} B_1 + 2\, R^{(0)^2} B] \sin\tau - [2\, R^{(0)^{1/2}} A_1 - 2\, R^{(0)^2} A] \cos\tau \\ &+ 2\, R^{(0)^2} H^{(2)} - 8\, R^{(0)^5} \quad . \end{aligned} \tag{43}$$

The solution, from which the secular τ-terms are removed, is

$$r^{(2)} = E \sin\tau + F \cos\tau - 8\, R^{(0)^5} + 2\, R^{(0)^2} H^{(2)} \quad . \tag{44}$$

The removal of the secular τ-terms of the solution of equation (43) gives the following differential equations for the functions $A(\tau_\epsilon)$ and $B(\tau_\epsilon)$:

$$\begin{aligned} B_1 &= R^{(0)^{3/2}} B \\ A_1 &= R^{(0)^{3/2}} A \end{aligned} \tag{45}$$

with boundary conditions

$$B(0) = 0$$
$$A(0) = -2 \qquad (46)$$

derived from equations (25) and (26) with

$$r^{(1)}(0,0) = u^{(1)}(0,0) = 0 \quad .$$

The solution of equations (45) and (46) gives

$$A = -2\, {R^{(0)}}^{1/2}$$
$$B = 0 \qquad (47)$$

The component of the velocity in radius direction $u^{(2)}$ follows as

$$u^{(2)} = [-2\, {R^{(0)}}^{2} - {R^{(0)}}^{-1/2} F] \sin\tau + [{R^{(0)}}^{-1/2} E] \cos\tau \quad , \qquad (48)$$

and

$$t^{(2)} = [{R^{(0)}}^{1/2} F - 4\, {R^{(0)}}^{3}] \sin\tau - {R^{(0)}}^{1/2} E \cos\tau + \frac{7}{2}\, {R^{(0)}}^{11/2} \tau^3 + T^{(2)}(\tau_\epsilon) \ ; \qquad (49)$$

the removed secular terms show that $T^{(1)}$ is dependent on $H^{(2)}$ as

$$T_1^{(1)} = -8\, {R^{(0)}}^{5} + 2\, {R^{(0)}}^{2} H^{(2)} \quad . \qquad (50)$$

The second order term of the central angle is derivable from the second equation of system (33)

$$\varphi^{(2)} = [2\, {R^{(0)}}^{3/2} - 4\, {R^{(0)}}^{2} - R^{(0)} F] \sin\tau + R^{(0)} E \cos\tau - {R^{(0)}}^{3/2} \sin 2\tau + \phi^{(2)}(\tau_\epsilon) \qquad (51)$$

and similar to equation (50) follows

$$\phi_1^{(1)} = 8\, R^{(0)^{1/2}} + 2\, R^{(0)} - 2\, R^{(0)^{5/2}}\, H^{(2)} + R^{(0)^{-1/2}}\, V^{(2)} \quad . \tag{52}$$

Equations (50) and (52) show that the derivation of a complete first order solution is dependent on $H^{(2)}$ and $V^{(2)}$.

D. Determination of Second Order Constants

The unknown constants of the second order solution are E, F, $H^{(2)}$, $V^{(2)}$, $\phi^{(2)}$, and $T^{(2)}$. Furthermore, $H^{(2)}$ and $V^{(2)}$ determine $\phi^{(1)}$ and $T^{(1)}$ from the first order solution. To determine these constants, it is not necessary to solve the third order equations completely; we need only to determine the secular τ-terms of this solution.

The third order differential equations are

$$\begin{aligned}
\omega\, r_2^{(3)} &= -\, r_1^{(2)} + u^{(3)} \\
\omega\, \varphi_2^{(3)} &= -\, \varphi_1^{(2)} + R^{(0)^{-1}}\, v^{(3)} - R^{(0)^{-3/2}}\, r^{(3)} + 2\, R^{(0)^{-5/2}}\, r^{(1)}\, r^{(2)} - R^{(0)^{-2}}\, v^{(2)}\, r^{(1)} \\
\omega\, u_2^{(3)} &= -\, u_1^{(2)} + 2\, H^{(0)}\, r^{(3)} + 2\, R^{(0)}\, h^{(3)} + 2\, h^{(2)}\, r^{(1)} + R^{(0)^{3/2}}\, u^{(2)} \\
&\quad + 2\, R^{(0)^{1/2}}\, r^{(1)}\, u^{(1)} \\
\omega\, v_2^{(3)} &= -\, v_1^{(2)} - \frac{1}{2}\, R^{(0)}\, u^{(1)^2} + 2\, R^{(0)}\, r^{(2)} + r^{(1)^2} \\
\omega\, h_2^{(3)} &= -\, h_1^{(2)} + v^{(2)} + \frac{1}{2}\, R^{(0)^{-1/2}}\, u^{(1)^2} \\
\omega\, t_2^{(3)} &= -\, t_1^{(2)} + r^{(3)}
\end{aligned} \tag{53}$$

The boundary conditions for equations (53) are

$$\begin{aligned}
r^{(3)}(0,0) &= 0 \\
\varphi^{(3)}(0,0) &= 0 \\
u^{(3)}(0,0) &= 0
\end{aligned} \tag{54}$$

$$v^{(3)}(0,0) = 0$$
$$h^{(3)}(0,0) = 0 \quad . \qquad (54)\ \text{(Concluded)}$$
$$t^{(3)}(0,0) = 0$$

Considering the energy equation we get with

$$u^{(1)^2} = 4\cos^2\tau - 8\,R^{(0)^{5/2}}\cos\tau + 4\,R^{(0)^5} \quad ,$$
$$h_1^{(2)} = H_1^{(2)} \qquad (55)$$

the third order energy term,

$$h^{(3)} = \frac{1}{2}\sin 2\tau + H^{(3)}(\tau_\epsilon) \quad , \qquad (56)$$

and collecting improper secular τ-terms yields

$$H_1^{(2)} = V^{(2)} + R^{(0)^{-1/2}} + 2\,R^{(0)^{9/2}} \quad . \qquad (57)$$

Similarly, the equation for $v^{(3)}$ gives for the secular part,

$$V_1^{(2)} = -\,R^{(0)} - 2\,R^{(0)^6} - 16\,R^{(0)^6} + 4\,R^{(0)^3}H^{(2)} + 2\,R^{(0)} \quad . \qquad (58)$$

Differentiating equation (57) gives a second order differential equation for $H^{(2)}$,

$$H_{11}^{(2)} - 4\,R^{(0)^3}H^{(2)} = 0 \quad . \qquad (59)$$

With the boundary conditions derived from equations (34), (35), and (36),

$$H^{(2)}(0) = 0$$
$$H_1^{(2)}(0) = -\,1 \quad , \qquad (60)$$

the solution of equation (59) is

$$H^{(2)} = C\, R^{(0)^{-2}} + K\, R^{(0)^{-1/2}} \quad . \tag{61}$$

With the boundary conditions (60) we will find

$$\begin{aligned} C &= \frac{1}{5} \\ K &= -\frac{1}{5} \end{aligned} \tag{62}$$

Consequently it follows that

$$H^{(2)} = \frac{1}{5} \left[- R^{(0)^{-1/2}} + R^{(0)^{-2}} \right] \quad , \tag{63}$$

and from equation (54) it follows that

$$V^{(2)} = -\frac{4}{5} R^{(0)^{-1/2}} - \frac{1}{5} R^{(0)^{2}} - 2\, R^{(0)^{9/2}} - R^{(0)^{1/2}} \quad . \tag{64}$$

With a knowledge of $H^{(2)}$ and $V^{(2)}$ a complete first order solution is known. To determine E and F, derive from the radius equation of equation (53), as similarly done for $r^{(2)}$,

$$r_{22}^{(3)} = - R^{(0)^{1/2}} r_{12}^{(2)} + u_2^{(2)} R^{(0)^{1/2}} \tag{65}$$

with

$$r_{12}^{(2)} = E_1 \cos\tau - F_1 \sin\tau \quad , \tag{66}$$

and $u_2^{(3)}$ defined by the third equation of equation (53) gives

$$r_{22}^{(3)} + r^{(3)} = - R^{(0)^{1/2}} (E_1 \cos\tau - F_1 \sin\tau)$$

$$- R^{(0)} [- 8 R^{(0)^{7/2}} + R^{(0)} F - R^{(0)^{-1/2}} F_1] \sin\tau$$

$$+ [- R^{(0)} E + R^{(0)^{-1/2}} E_1] \cos\tau + 2 R^{(0)^2} h^{(3)}$$

$$+ 2 R^{(0)} h^{(2)} r^{(1)} + R^{(0)^{5/2}} u^{(2)} + 2 R^{(0)^{3/2}} r^{(1)} u^{(1)} \quad . \tag{67}$$

Collecting secular terms of τ after integration of (67) gives

$$E_1 = R^{(0)^{3/2}} E \tag{68}$$

or

$$E = CR^{(0)^{1/2}} \quad , \tag{69}$$

and from equation (48), using

$$h^{(2)}(0,0) = 0 \quad , \quad C = 0 \quad , \tag{70}$$

we obtain

$$E = 0 \quad . \tag{71}$$

For F we will find

$$F_1 = R^{(0)^{3/2}} F + R^{(0)^4} + 2 R^{(0)} H^{(2)} \tag{72}$$

with the solution using equation (63) being

$$F = \frac{1}{4} R^{(0)^{5/2}} + \frac{2}{5} - \frac{1}{15} R^{(0)^{-7/2}} + R^{(0)^{1/2}} \frac{85}{12} \quad , \tag{73}$$

whereby the boundary condition,

$$F(0) = 8 \quad , \tag{74}$$

is derived from equation (44) with the use of $r^{(2)}(0,0) = 0$.

From equations (50), (52), (63), and (64) we find, obeying boundary conditions,

$$T^{(1)} = \frac{8}{7}\left[1 - R^{(0)^{7/2}}\right] - \frac{2}{25}\left[1 - R^{(0)^{5/2}}\right] + \frac{1}{5}\left[1 - R^{(0)}\right] - 2 \tag{75}$$

and

$$\phi^{(1)} = -\frac{4}{25} R^{(0)^{-5/2}} - \frac{1}{5} R^{(0)^{-3/2}} - \frac{1}{5} R^{(0)^{-1}} - 2 R^{(0)^{-1/2}} - \frac{2}{15} R^{(0)^{3/2}} - \frac{2}{5} R^{(0)^{-5/2}}$$

$$+ R^{(0)^{4}} - \frac{1}{10} \ln\left(R^{(0)}\right) + \frac{7}{75} \quad . \tag{76}$$

To derive a complete second order solution for the time t and the central angle φ, it would be necessary to evaluate a complete third order solution from equations (53) and to determine the unknown functions out of the fourth order differential equations. However, looking at the solution so far derived, it seems reasonable to neglect the functions $T^{(2)}$ and $\phi^{(2)}$ in comparison with the unbounded terms of τ. $T^{(2)}$ and $\phi^{(2)}$ are only chosen to fulfill the boundary conditions

$$\phi^{(2)} = 0$$

and

$$T^{(2)} = 0 \quad .$$

E. Summarized Solution

In this subsection, the derived solution is collected. The independent variables are calculated as

$$\tau_\epsilon = \epsilon s$$

$$\tau = \frac{1}{4\epsilon}\,[1 - R^{(0)^{-2}}] \tag{77}$$

with

$$\epsilon = \frac{F}{m} = \text{constant} \quad , \tag{78}$$

and

$$R^{(0)} = (1 - 3\,\tau_\epsilon)^{-2/3} \tag{79}$$

is the base solution for the radius of the spiral orbit. The variables of the orbit are approximately

$$r = R^{(0)} + \epsilon\,[-2\,R^{(0)^{1/2}} \sin\tau] + \epsilon^2 \left\{F \cos\tau - 8\,R^{(0)^5} - \frac{2}{5}\,[R^{(0)^{5/2}} - 1]\right\} .$$

$$u = \epsilon\,[-2\cos\tau + 2\,R^{(0)^{5/2}}] + \epsilon^2 \left\{[-R^{(0)^{-1/2}} F - 2\,R^{(0)^2}] \sin\tau\right\}$$

$$v = R^{(0)^{1/2}} + \epsilon^2\,[4\,R^{(0)^2} \cos\tau - \frac{4}{5}\,R^{(0)^{-1/2}} - R^{(0)^{1/2}} - \frac{1}{5}\,R^{(0)^2} - 2\,R^{(0)^{9/2}}]$$

$$h = -\,[2R^{(0)}]^{-1} + \epsilon^2 \left[\frac{1}{5}\,R^{(0)^{-2}} - \frac{1}{5}\,R^{(0)^{1/2}}\right]$$

$$\varphi = \tau + \epsilon\,[2\,R^{(0)^{-1/2}} \cos\tau + \phi^{(1)}] + \epsilon^2 \left\{ [2\,R^{(0)^{3/2}} - 4\,R^{(0)^2} - R^{(0)} F] \sin\tau - R^{(0)^{3/2}} \sin 2\tau \right\} \tag{80}$$

$$t = R^{(0)^{3/2}}\,\tau + \epsilon\,[-3\,R^{(0)^{7/2}}\,\tau^2/2 + 2\,R^{(0)} \cos\tau + T^{(1)}] + \epsilon^2 \left\{\frac{7}{2}\,R^{(0)^{11/2}}\,\tau^3 + [R^{(0)^{1/2}} F - 4\,R^{(0)^3}] \sin\tau\right\}$$

where $T^{(1)}$ and $\phi^{(1)}$ are defined by equations (75) and (76) and F is defined by equation (73).

IV. AN ALTERNATE METHOD TO DETERMINE T

In the case ϵ = constant as considered here, we know from equation (1) that

$$t = \int_0^s r(s)\,ds \quad . \tag{81}$$

Equation (81) is an independent equation. Hence, with the approximate solution for $r[\tau_\epsilon(s),\tau(s)]$ derived, we will be able to find t as follows:

$$t \approx \int_0^s R^{(0)} ds - 2\epsilon \int_0^s {R^{(0)}}^{1/2} \sin\tau\, ds \quad . \tag{82}$$

With equations (23), (31), and (32) we determine the derivatives

$$\frac{dR^{(0)}}{ds} = 2\,\epsilon\; {R^{(0)}}^{5/2} \tag{83}$$

and

$$\frac{dt}{dR^{(0)}} = \frac{1}{2\,\epsilon\, {R^{(0)}}^{3}} \quad , \tag{84}$$

and in a straightforward manner, we solve the first integral from equation (82),

$$\int R^{(0)} ds = \frac{1}{\epsilon}\,[1 - {R^{(0)}}^{-1/2}] \quad . \tag{85}$$

From two successive partial integrations of the second integral of equation (82), we determine

$$-2\epsilon \int_0^s {R^{(0)}}^{1/2} \sin\tau\, ds = 2_\epsilon\,[R^{(0)} \cos\tau - 1] - 4\epsilon^2\, {R^{(0)}}^3 \sin\tau + 0(\epsilon^3) \tag{86}$$

or

$$t = \frac{1}{\epsilon} [1 - R^{(0)^{-1/2}}] + 2\epsilon [R^{(0)} \sin \tau - 1] - 4\epsilon^2 R^{(0)^3} \sin \tau + 0(\epsilon^3) \quad . \tag{87}$$

In numerical investigations, it was found that t determined with equation (87) is much more accurate than t calculated with the aid of equations (80).

V. SOLUTION ANALYSIS

A. Solution Characteristics

Because of the regularizing transformation, the relationship between the velocity ${}^{s}v$ calculated dependent on s and the natural velocity ${}^{t}v$ dependent on t is

$${}^{s}v = v(s) = rv(t) = r\,{}^{t}v \tag{88}$$

Obeying equation (88), the zero order solution derived corresponds to the standard circular asymptotic solution of spiral type trajectories,

$$r^{(0)}(\tau_\epsilon,\tau) \cdot {}^{t}v^{(0)^2}(\tau_\epsilon,\tau) = 1 \quad . \tag{89}$$

As seen from the differential equations of motion (1), the thrusting terms are of the first order. It follows therefrom that the zero order solution will also match the expansion solution of the optimal thrust trajectory. The difference between tangential and optimal thrust appears in the first order terms which show similar characteristics; i.e., oscillations with slowly varying phase, frequency, and amplitude.

The formulation of the equations of motion used causes all variables to steadily increase in value with superimposed oscillation. Figure 1 shows as an example for $\epsilon = 0.001$ the behavior of the radius r and the energy h over the first few revolutions of the trajectory. The regularizing transformation corresponds to a scaling of the time history of the motion. Figure 2 shows the relationship between the regularized scaled time and the physically scaled time for $\epsilon = 0.001$. Using the energy h as an additional state variable simplifies the analysis of the expansion considerably.

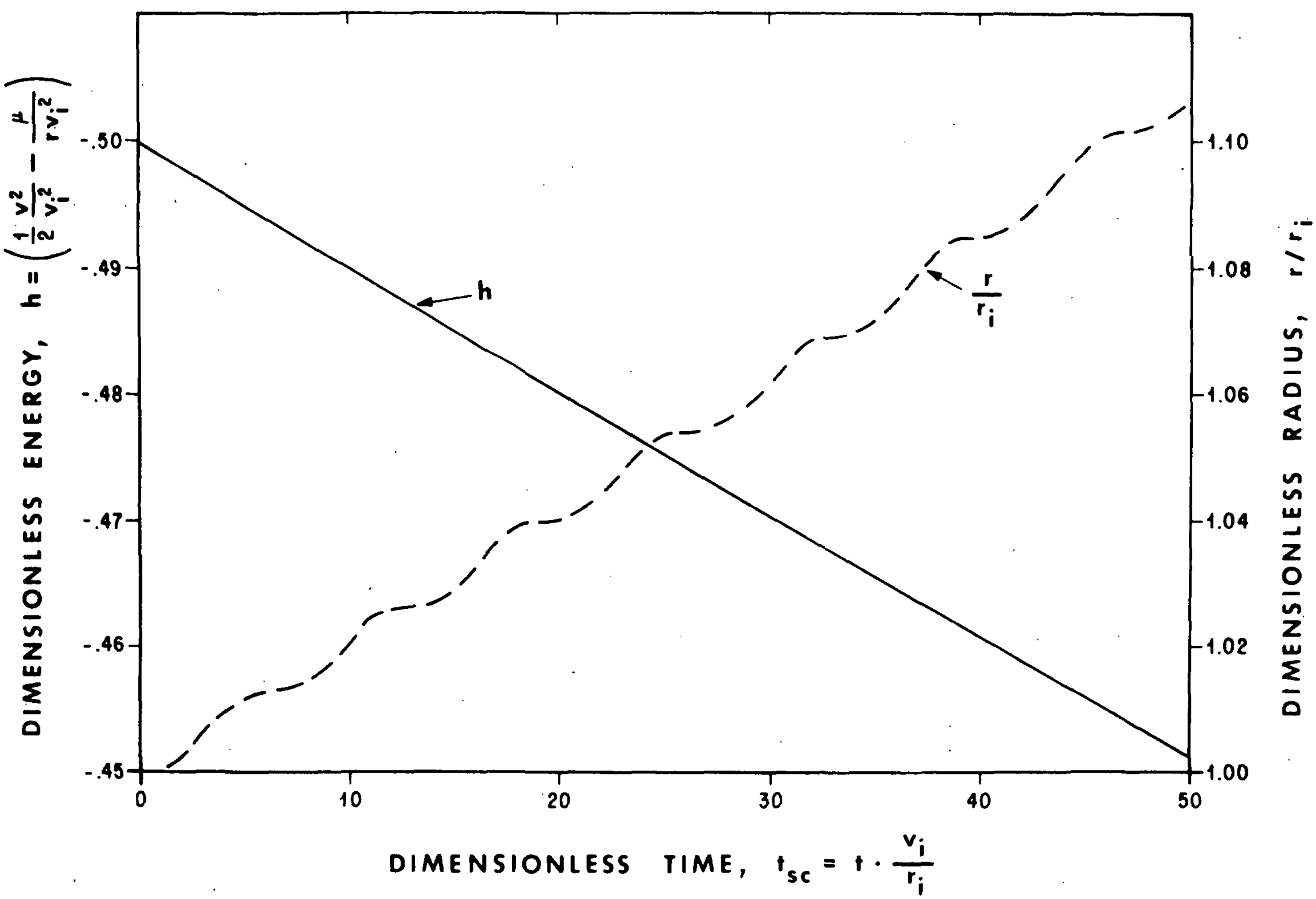

Figure 1. Radius and energy behavior for the first seven orbits with $\epsilon = 0.001$.

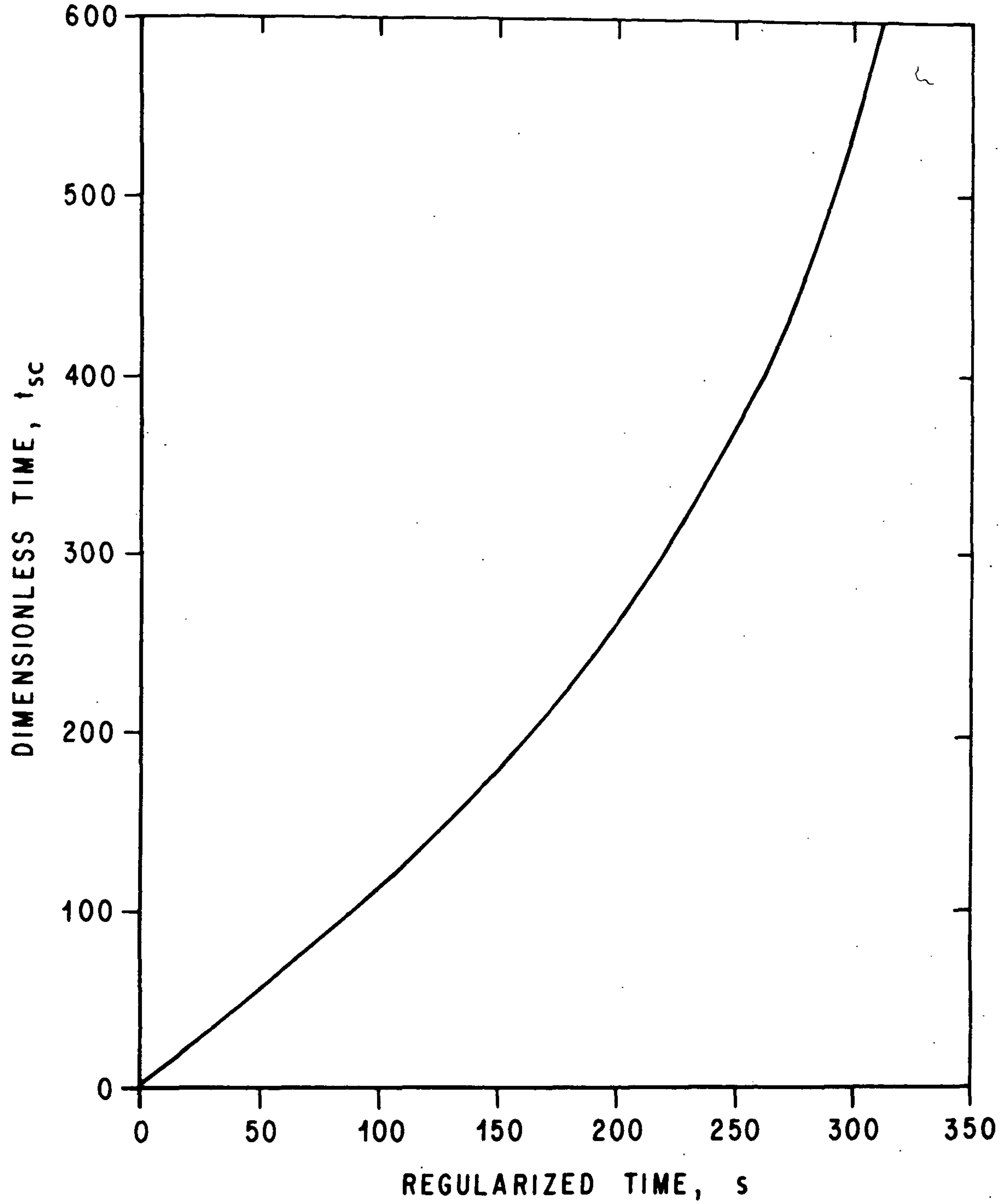

Figure 2. Dimensionless time versus regularized time for $\epsilon = 0.001$.

B. Solution Limitations

The low thrust solution derived has a singularity at

$$\tau_\epsilon = \frac{1}{3} \quad . \tag{90}$$

This value corresponds to the escape condition,

$$h = 0 \quad , \tag{91}$$

and leads to

$$r\left(\tau_\epsilon = \frac{1}{3}\right) \to \infty \tag{92}$$

which corresponds to the Keplerian solution for the equations of motion without thrust. This solution is, however, physically unacceptable; consequently, the solution presented is valid for energy levels less than zero only. Near escape, the thrusting acceleration and the gravity acceleration terms are of the same order of magnitude, and the motion ceases to possess two characteristic time scales. This changing in motion characteristics is because of the failure of the expansion presented which implies the assumption of the dominance of the gravity force. A complete solution of the singular escape problem would require the matching of the solution presented with an expansion of a different type in the vicinity of escape conditions.

In addition, the expansion presented assumes that the component of the velocity in the radius direction u is small in comparison to the component in the circumferential direction v. Again, near escape, this condition is not fulfilled, and the expansion solution fails to describe the trajectory.

C. Numerical Comparisons

To check the validity of the second order two-variable expansion, a comparison with numerically generated trajectories with different ϵ-values was made. The terminal energy was always chosen as $h_f = 0$ to explore the accuracy limits of the solution derived for each ϵ-value. In general, the accuracy of the solution increases with decreasing ϵ-values. In other words, for a prespecified accuracy, the validity range of the approximation will be extended in the direction of escape conditions with decreasing ϵ-values.

Because of the high accuracy up to a certain energy level, no difference between both solutions would be noticeable in a diagram (such as Figures 1 and 2) showing a comparison of the analytic approximation with the numerically calculated trajectory. To make accuracy and limitations of the two-variable expansion solutions clear, Figure 3 shows the validity limits for various ϵ-values. These limits depend on the accuracy required, expressed in number of significance digits of the most inaccurate variable u. Figure 3 shows how many digits of the analytic approximation of u coincide with the numerically generated solution. For the numerical integration, a Runge-Kutta-Fehlberg [7] formula of the seventh order with stepsize control was used.

Figure 4 shows the corresponding numbers of revolutions around the central body. Figures 3 and 4 show that for $\epsilon \leqslant 0.001$ the high accuracy of the approximate solution will be kept over the whole inner multirevolution part of the spiral trajectories. The loss of accuracy occurs in the outer part, when the variables change more rapidly. The accuracy in φ and t corresponds to the accuracy of u. The accuracy in r is about 1 to 2, and in h and v is about three to four orders of magnitude better than the accuracy of u.

A numerically generated comparison for the example $\epsilon = 0.001$ is given in computer printout form in the Appendix. s is the independent variable; $x = x(r, \varphi, u, v, h, t)$, where x is the numerically generated state vector; and $y = y(r, \varphi, u, v, h, t)$, where y is the approximate solution.

VI. CONCLUDING REMARKS

An analytical approximate solution is presented for a low-thrust energy increase trajectory whereby a constant acceleration acting in a tangential direction was assumed. The solution was derived using a two-variable expansion method. The solution is shown to be very accurate for values of the disturbing acceleration $\epsilon \leqslant 0.001$. This accuracy holds up to high energy levels, close to escape.

The solution fails to describe the trajectory near escape because the method of derivation of the presented solution assumes the gravity force to be strong in comparison to a small disturbing force. Near escape, thrust and gravity acceleration are of the same order of magnitude. Furthermore, the solution implies that the velocity component in the circumferential direction is large compared to the component in the radius direction — a condition that is not valid near escape.

Even considering those limitations, the solution presented offers a very good simplified problem to gain experience in evaluating an accurate analytical approximation of the optimal low thrust trajectory. The tangential thrust trajectory solution keeps most of the characteristics of the solution for the optimal trajectory. The differences between those appear in the first order terms.

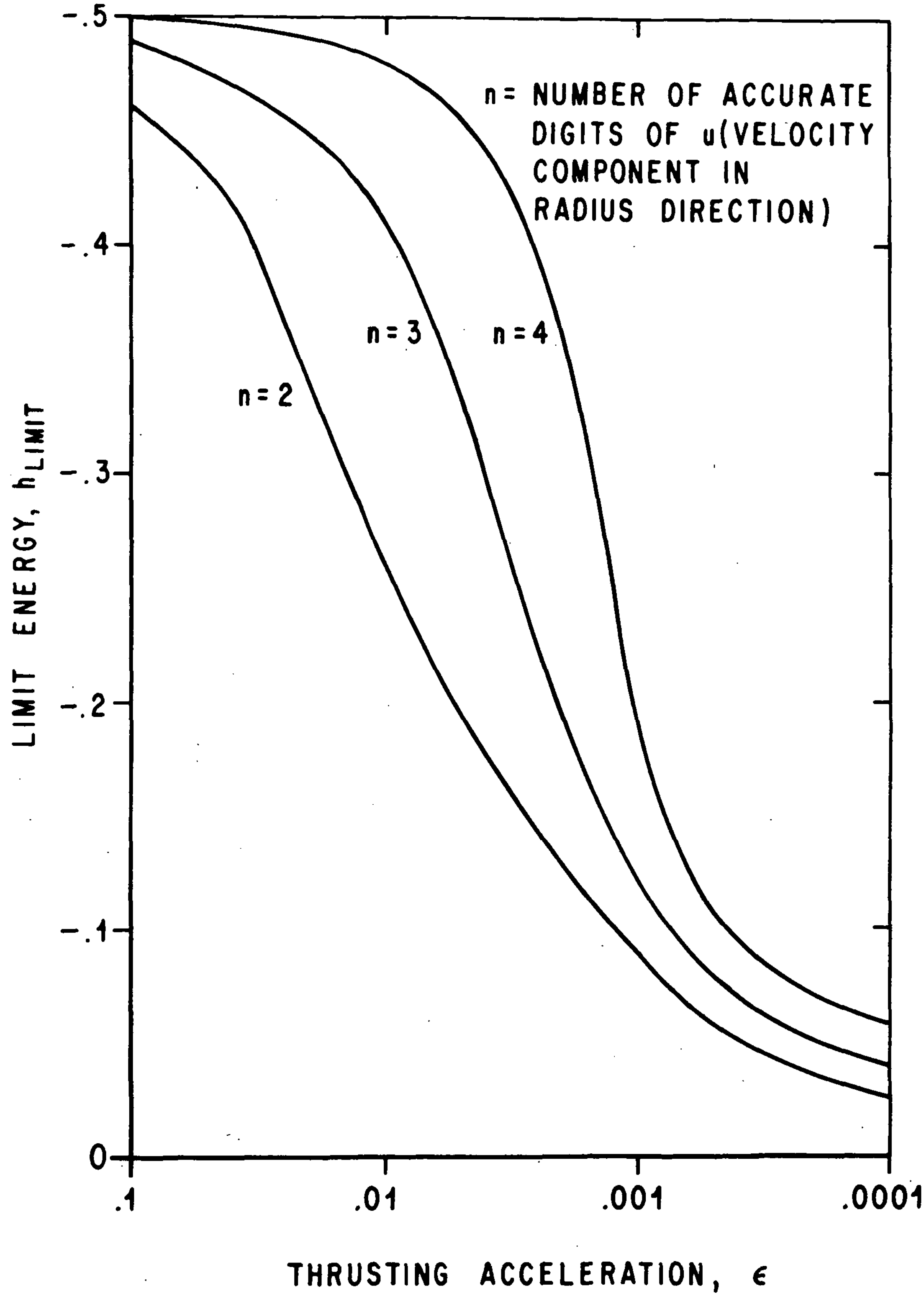

Figure 3. Energy level validity limits of the two-variable expansion solution.

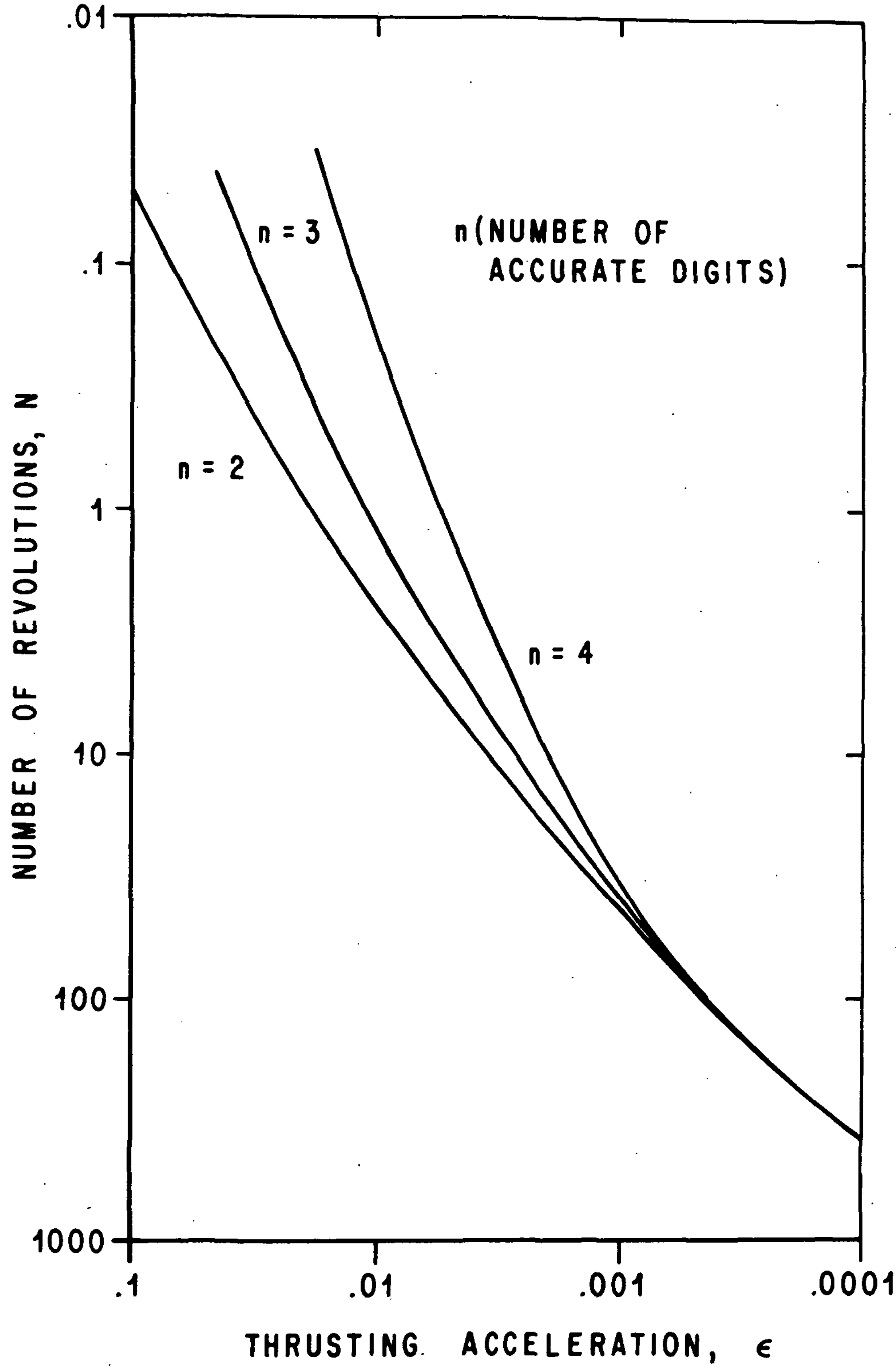

Figure 4. Number of revolution validity limits for the two-variable expansion solution.

APPENDIX

TABULATED COMPARISON OF NUMERICALLY CALCULATED ORBITS AND THE TWO-VARIABLE ANALYTICAL APPROXIMATED SOLUTION

x(r, φ, u, v, h, t) NUMERICALLY GENERATED STATE VECTOR
y(r, φ, u, v, h, t) ANALYTICAL SOLUTION

EPS .10000000E 00 BETA .00000000E 00

S	.00000000E 00					
X	.10000000E 01	.00000000E 00	.00000000E 00	.10000000E 01	-.50000000E 00	.00000000E 00
S	.30000000E-01					
X	.10000009E 01	.30044993E-01	.90173203E-04	.10030000E 01	-.49699550E 00	.30000007E-01
Y	.99454818E 00	-.23893681E-03	.22262738E-03	.10029048E 01	-.49696548E 00	.29989384E-01
S	.15000000E 00					
X	.10001132E 01	.15112073E 00	.22683091E-02	.10150008E 01	-.48488749E 00	.15000424E 00
Y	.98526513E 00	.11635387E 00	.36134546E-02	.10144066E 01	-.48473517E 00	.14975665E 00
S	.63000000E 00					
X	.10084421E 01	.64848778E 00	.40147137E-01	.10632572E 01	-.43500257E 00	.63132931E 00
Y	.93614114E 00	.55412065E 00	.60080003E-01	.10573754E 01	-.43419569E 00	.62826435E 00
S	.25500000E 01					
X	.15192566E 01	.25851309E 01	.59683447E 00	.13231689E 01	-.20179146E 00	.28882203E 01
Y	-.76747261E 01	.60126494E 01	.21494874E 01	-.10360368E 00	-.18745564E 00	.27359633E 01
S	.25579620E 01					
X	.52766016E 01	.39829007E 01	.48055317E 01	.25100649E 01	.33536163E 00	.87212266E 01
Y	-.80031973E 01	.61426749E 01	.21875138E 01	-.14945334E 00	-.18614823E 00	.27421701E 01
S	.25579620E 01					
X	.52766016E 01	.39829007E 01	.48055317E 01	.25100649E 01	.33536163E 00	.87212266E 01
Y	-.80031973E 01	.61426749E 01	.21875138E 01	-.14945334E 00	-.18614823E 00	.27421701E 01

EPS .10000000E-01 BETA .00000000E 00

S	.00000000E 00					
X	.10000000E 01	.00000000E 00	.00000000E 00	.10000000E 01	-.50000000E 00	.00000000E 00
S	.15000000E 00					
X	.10000112E 01	.15011208E 00	.22480403E-03	.10015000E 01	-.49849887E 00	.15000042E 00
Y	.99996783E 00	.14665871E 00	.23059216E-03	.10014995E 01	-.49849872E 00	.15000060E 00
S	.25500000E 01					
X	.10408298E 01	.25540374E 01	.38283352E-01	.10260796E 01	-.47416661E 00	.25788933E 01
Y	.10407994E 01	.24808622E 01	.38479641E-01	.10260656E 01	-.47416075E 00	.25805566E 01
S	.89500000E 01					
X	.12116137E 01	.85609302E 01	.44862962E-01	.11087753E 01	-.40593528E 00	.98501518E 01
Y	.12115569E 01	.85035013E 01	.45223489E-01	.11087434E 01	-.40591638E 00	.98613815E 01
S	.15350000E 02					
X	.14787448E 01	.14056588E 02	.51308854E-01	.12270431E 01	-.33137416E 00	.18540023E 02
Y	.14784694E 01	.14043776E 02	.52360724E-01	.12269889E 01	-.33134235E 00	.18574770E 02
S	.24310000E 02					
X	.23098730E 01	.20708252E 02	.16282922E 00	.15362626E 01	-.20927044E 00	.35075743E 02
Y	.22970954E 01	.20912184E 02	.17858751E 00	.15350006E 01	-.20920494E 00	.35276223E 02
S	.33270000E 02					
X	.75562739E 01	.25519123E 02	.19754687E 01	.29744417E 01	-.20690730E-01	.70172518E 02
Y	-.94162979E 06	.18060702E 06	.68629650E 03	-.29149096E 05	-.75087105E-02	.10419652E 03
S	.34488779E 02					
X	.12704590E 02	.26092464E 02	.51485160E 01	.40872541E 01	.55151462E-01	.85692185E 02

FT1 < 0

S	.34488779E 02					
X	.12704590E 02	.26092464E 02	.51485160E 01	.40872541E 01	.55151462E-01	.85692185E 02

FT1 < 0

EPS .10000000E-02 BETA .00000000E 00

S	.00000000E 00					
X	.10000000E 01	.00000000E 00	.00000000E 00	.10000000E 01	-.50000000E 00	.00000000E 00
S	.25500000E 01					
X	.10039943E 01	.25504083E 01	.36764496E-02	.10025557E 01	-.49744674E 00	.25528470E 01
Y	.10039945E 01	.25428166E 01	.36767981E-02	.10025557E 01	-.49744674E 00	.25528630E 01
S	.13430000E 02					
X	.10263750E 01	.13339664E 02	.70298281E-03	.10138012E 01	-.48647818E 00	.13613873E 02
Y	.10263765E 01	.13338274E 02	.70288954E-03	.10138012E 01	-.48647815E 00	.13613983E 02
S	.26230000E 02					
X	.10547537E 01	.25880514E 02	.81948167E-03	.10276949E 01	-.47341341E 00	.26949075E 02
Y	.10547582E 01	.25879246E 02	.81856880E-03	.10276949E 01	-.47341335E 00	.26949312E 02
S	.39030000E 02					
X	.10854656E 01	.38248110E 02	.74985290E-03	.10423834E 01	-.46016579E 00	.40660369E 02
Y	.10854731E 01	.38247364E 02	.74932057E-03	.10423834E 01	-.46016574E 00	.40660752E 02
S	.51830000E 02					
X	.11188808E 01	.50437647E 02	.67604815E-03	.10579495E 01	-.44672465E 00	.54775235E 02
Y	.11188905E 01	.50437470E 02	.67831054E-03	.10579494E 01	-.44672455E 00	.54775787E 02
S	.64630000E 02					
X	.11553226E 01	.62444145E 02	.10097124E-02	.10744880E 01	-.43307815E 00	.69324275E 02
Y	.11553317E 01	.62443830E 02	.10174146E-02	.10744878E 01	-.43307802E 00	.69325022E 02
S	.79990000E 02					
X	.11986612E 01	.76602589E 02	.24404953E-02	.10957748E 01	-.41641265E 00	.87405286E 02
Y	.11986711E 01	.76600305E 02	.24294198E-02	.10957747E 01	-.41641249E 00	.87406324E 02
S	.94070000E 02					
X	.12451717E 01	.89331833E 02	.30700719E-02	.11168594E 01	-.40083753E 00	.10463417E 03
Y	.12451799E 01	.89329178E 02	.30557833E-02	.11168592E 01	-.40083734E 00	.10463552E 03
S	.10943000E 03					
X	.13022396E 01	.10293718E 03	.53658243E-02	.11418285E 01	-.38349350E 00	.12421648E 03
Y	.13022277E 01	.10293151E 03	.53517620E-02	.11418283E 01	-.38349328E 00	.12421826E 03
S	.12607000E 03					
X	.13747249E 01	.11732631E 03	.53450668E-02	.11716027E 01	-.36424979E 00	.14647267E 03
Y	.13747122E 01	.11732209E 03	.53624689E-02	.11716023E 01	-.36424954E 00	.14647500E 03

S	.14271000E 03					
X	.14527757E 01	.13133354E 03	.34354788E-02	.12047553E 01	-.34448348E 00	.16995713E 03
Y	.14527974E 01	.13133422E 03	.34503739E-02	.12047549E 01	-.34448319E 00	.16996022E 03
S	.16319000E 03					
X	.15665349E 01	.14802413E 03	.79988038E-02	.12512542E 01	-.31934567E 00	.20080924E 03
Y	.15665086E 01	.14802020E 03	.80086739E-02	.12512537E 01	-.31934533E 00	.20081362E 03
S	.18367000E 03					
X	.17036772E 01	.16405179E 03	.60536905E-02	.13059185E 01	-.29317576E 00	.23425859E 03
Y	.17037036E 01	.16405495E 03	.60395689E-02	.13059181E 01	-.29317538E 00	.23426494E 03
S	.20415000E 03					
X	.18818748E 01	.17936837E 03	.11614775E-01	.13715395E 01	-.26577993E 00	.27090381E 03
Y	.18818447E 01	.17936952E 03	.11629132E-01	.13715387E 01	-.26577950E 00	.27091318E 03
S	.22463000E 03					
X	.21125994E 01	.19388963E 03	.11679114E-01	.14527722E 01	-.23688989E 00	.31166983E 03
Y	.21126300E 01	.19390125E 03	.11707863E-01	.14527711E 01	-.23688940E 00	.31168439E 03
S	.24511000E 03					
X	.24246159E 01	.20752531E 03	.16339709E-01	.15574263E 01	-.20611380E 00	.35792925E 03
Y	.24246589E 01	.20755025E 03	.16357133E-01	.15574254E 01	-.20611324E 00	.35795367E 03
S	.26559000E 03					
X	.28905812E 01	.22014042E 03	.26441597E-01	.17006608E 01	-.17283407E 00	.41202264E 03
Y	.28906210E 01	.22019666E 03	.26511027E-01	.17006590E 01	-.17283343E 00	.41206869E 03
S	.28607000E 03					
X	.36754811E 01	.23153588E 03	.50376081E-01	.19170534E 01	-.13595702E 00	.47844210E 03
Y	.36754295E 01	.23169608E 03	.50761797E-01	.19170425E 01	-.13595627E 00	.47854933E 03
S	.30655000E 03					
X	.53349509E 01	.24137525E 03	.13144845E 00	.23136286E 01	-.93103107E-01	.56808029E 03
Y	.53321902E 01	.24215872E 03	.13532012E 00	.23134588E 01	-.93101883E-01	.56847423E 03
S	.33089886E 03					
X	.17630588E 02	.24996261E 03	.19458153E 01	.43939410E 01	-.19571727E-01	.78311760E 03
Y	-.79267578E 02	.74768983E 03	.72765757E 01	.19762937E-01	-.18820473E-01	.80604859E 03
S	.33089886E 03					
X	.17630588E 02	.24996261E 03	.19458153E 01	.43939410E 01	-.19571727E-01	.78311760E 03
Y	-.79267578E 02	.74768983E 03	.72765757E 01	.19762937E-01	-.18820473E-01	.80604859E 03

EPS .10000000E-03 BETA .00000000E 00

S	.00000000E 00					
X	.10000000E 01	.00000000E 00	.00000000E 00	.10000000E 01	-.50000000E 00	.00000000E 00
S	.13430000E 02					
X	.10025438E 01	.13421043E 02	.70024391E-04	.10013466E 01	-.49865610E 00	.13448008E 02
Y	.10025440E 01	.13420876E 02	.69973644E-04	.10013466E 01	-.49865610E 00	.13448009E 02
S	.34550000E 02					
X	.10069570E 01	.34490578E 02	.40334576E-03	.10034790E 01	-.49653900E 00	.34669658E 02
Y	.10069566E 01	.34489752E 02	.40304351E-03	.10034790E 01	-.49653900E 00	.34669661E 02
S	.56310000E 02					
X	.10115005E 01	.56150879E 02	.20862323E-04	.10056953E 01	-.49435303E 00	.56630074E 02
Y	.10115009E 01	.56150820E 02	.21415671E-04	.10056953E 01	-.49435303E 00	.56630078E 02
S	.77430000E 02					
X	.10155926E 01	.77128904E 02	.24064453E-03	.10078651E 01	-.49222671E 00	.78037160E 02
Y	.10155930E 01	.77128415E 02	.23968764E-03	.10078651E 01	-.49222671E 00	.78037167E 02
S	.97910000E 02					
X	.10200826E 01	.97427918E 02	.41051676E-03	.10099871E 01	-.49016043E 00	.98884181E 02
Y	.10200817E 01	.97427099E 02	.41004697E-03	.10099871E 01	-.49016043E 00	.98884189E 02
S	.12223000E 03					
X	.10250429E 01	.12147712E 03	.31458615E-03	.10125305E 01	-.48770106E 00	.12375490E 03
Y	.10250429E 01	.12147650E 03	.31309171E-03	.10125305E 01	-.48770105E 00	.12375491E 03
S	.14783000E 03					
X	.10305412E 01	.14672667E 03	.33752735E-03	.10152357E 01	-.48510552E 00	.15007052E 03
Y	.10305409E 01	.14672601E 03	.33543367E-03	.10152357E 01	-.48510552E 00	.15007054E 03
S	.17343000E 03					
X	.10361066E 01	.17190866E 03	.34875701E-03	.10179699E 01	-.48250302E 00	.17652752E 03
Y	.10361060E 01	.17190799E 03	.34609452E-03	.10179699E 01	-.48250302E 00	.17652754E 03
S	.19903000E 03					
X	.10417388E 01	.19702274E 03	.34942298E-03	.10207339E 01	-.47989349E 00	.20312782E 03
Y	.10417381E 01	.19702207E 03	.34617908E-03	.10207339E 01	-.47989349E 00	.20312784E 03
S	.22463000E 03					
X	.10474398E 01	.22206851E 03	.33902514E-03	.10235282E 01	-.47727684E 00	.22987336E 03
Y	.10474393E 01	.22206788E 03	.33516986E-03	.10235282E 01	-.47727684E 00	.22987339E 03

S	.25023000E 03					
X	.10532144E 01	.24704562E 03	.31556425E-03	.10263532E 01	-.47465300E 00	.25676616E 03
Y	.10532144E 01	.24704504E 03	.31111699E-03	.10263532E 01	-.47465300E 00	.25676619E 03
S	.27583000E 03					
X	.10590701E 01	.27195368E 03	.27657379E-03	.10292098E 01	-.47202188E 00	.28380825E 03
Y	.10590713E 01	.27195318E 03	.27171011E-03	.10292098E 01	-.47202188E 00	.28380828E 03
S	.30143000E 03					
X	.10650187E 01	.29679230E 03	.22128498E-03	.10320984E 01	-.46938342E 00	.31100174E 03
Y	.10650215E 01	.29679192E 03	.21646666E-03	.10320984E 01	-.46938342E 00	.31100177E 03
S	.32319000E 03					
X	.10704683E 01	.31785048E 03	.40694699E-03	.10345794E 01	-.46713488E 00	.33423656E 03
Y	.10704625E 01	.31784974E 03	.40769248E-03	.10345794E 01	-.46713487E 00	.33423659E 03
S	.34879000E 03					
X	.10764986E 01	.34255972E 03	.44058399E-03	.10375290E 01	-.46448258E 00	.36171565E 03
Y	.10764925E 01	.34255893E 03	.43888819E-03	.10375290E 01	-.46448258E 00	.36171569E 03
S	.37439000E 03					
X	.10825964E 01	.36719838E 03	.43505545E-03	.10405126E 01	-.46182270E 00	.38935237E 03
Y	.10825915E 01	.36719761E 03	.43047766E-03	.10405126E 01	-.46182270E 00	.38935241E 03
S	.39871000E 03					
X	.10884603E 01	.39053911E 03	.13998121E-03	.10433791E 01	-.45928869E 00	.41575561E 03
Y	.10884664E 01	.39053892E 03	.13606979E-03	.10433791E 01	-.45928869E 00	.41575566E 03
S	.42431000E 03					
X	.10949381E 01	.41503896E 03	.62749173E-04	.10464308E 01	-.45661374E 00	.44370631E 03
Y	.10949459E 01	.41503892E 03	.62947780E-04	.10464308E 01	-.45661374E 00	.44370636E 03
S	.44991000E 03					
X	.11015560E 01	.43946712E 03	.62046457E-04	.10495185E 01	-.45393094E 00	.47182140E 03
Y	.11015622E 01	.43946708E 03	.67240508E-04	.10495185E 01	-.45393093E 00	.47182144E 03
S	.47551000E 03					
X	.11082472E 01	.46382319E 03	.16545740E-03	.10526430E 01	-.45124018E 00	.50010327E 03
Y	.11082479E 01	.46382296E 03	.17372894E-03	.10526430E 01	-.45124017E 00	.50010331E 03
S	.50239000E 03					
X	.11152031E 01	.48931874E 03	.11114575E-03	.10559642E 01	-.44840622E 00	.52998179E 03
Y	.11152073E 01	.48931863E 03	.11905053E-03	.10559642E 01	-.44840622E 00	.52998183E 03
S	.52927000E 03					
X	.11222675E 01	.51473375E 03	.83352105E-04	.10593276E 01	-.44556327E 00	.56005001E 03
Y	.11222742E 01	.51473371E 03	.90144689E-04	.10593276E 01	-.44556327E 00	.56005006E 03
S	.55487000E 03					
X	.11292716E 01	.53886333E 03	.24559698E-03	.10625711E 01	-.44284725E 00	.58886538E 03
Y	.11292692E 01	.53886298E 03	.25503175E-03	.10625711E 01	-.44284725E 00	.58886543E 03

S	.58175000E 03					
X	.11366085E 01	.56411956E 03	.22488814E-03	.10660200E 01	-.43998644E 00	.61931287E 03
Y	.11366073E 01	.56411926E 03	.23501651E-03	.10660200E 01	-.43998643E 00	.61931293E 03
S	.60863000E 03					
X	.11440683E 01	.58929370E 03	.22086006E-03	.10695140E 01	-.43711629E 00	.64995929E 03
Y	.11440678E 01	.58929342E 03	.23148539E-03	.10695140E 01	-.43711629E 00	.64995934E 03
S	.63551000E 03					
X	.11516568E 01	.61438520E 03	.23314824E-03	.10730544E 01	-.43423669E 00	.68080791E 03
Y	.11516558E 01	.61438491E 03	.24419202E-03	.10730544E 01	-.43423669E 00	.68080797E 03
S	.66239000E 03					
X	.11593751E 01	.63939353E 03	.26231715E-03	.10766420E 01	-.43134751E 00	.71186210E 03
Y	.11593727E 01	.63939319E 03	.27360723E-03	.10766420E 01	-.43134750E 00	.71186217E 03
S	.68927000E 03					
X	.11672201E 01	.66431811E 03	.30906046E-03	.10802782E 01	-.42844862E 00	.74312534E 03
Y	.11672152E 01	.66431770E 03	.32007743E-03	.10802782E 01	-.42844861E 00	.74312541E 03
S	.71743000E 03					
X	.11755118E 01	.69033894E 03	.15112856E-03	.10841408E 01	-.42540113E 00	.77610563E 03
Y	.11755184E 01	.69033884E 03	.16194576E-03	.10841408E 01	-.42540113E 00	.77610571E 03
S	.74559000E 03					
X	.11838627E 01	.71626667E 03	.10179980E-03	.10880592E 01	-.42234268E 00	.80932335E 03
Y	.11838763E 01	.71626667E 03	.10502398E-03	.10880592E 01	-.42234268E 00	.80932344E 03
S	.77247000E 03					
X	.11922339E 01	.74092836E 03	.12504370E-03	.10918528E 01	-.41941289E 00	.84125665E 03
Y	.11922449E 01	.74092833E 03	.13370726E-03	.10918528E 01	-.41941289E 00	.84125674E 03
S	.80319000E 03					
X	.12016438E 01	.76900794E 03	.48251508E-03	.10962539E 01	-.41605198E 00	.87802627E 03
Y	.12016343E 01	.76900729E 03	.47176426E-03	.10962539E 01	-.41605198E 00	.87802639E 03
S	.83263000E 03					
X	.12113811E 01	.79581075E 03	.42301391E-03	.11005391E 01	-.41281832E 00	.91354466E 03
Y	.12113700E 01	.79581021E 03	.43314541E-03	.11005391E 01	-.41281831E 00	.91354475E 03
S	.86079000E 03					
X	.12205884E 01	.82135000E 03	.35061633E-03	.11047014E 01	-.40971335E 00	.94778059E 03
Y	.12205827E 01	.82134960E 03	.36432117E-03	.11047014E 01	-.40971335E 00	.94778069E 03
S	.89023000E 03					
X	.12301965E 01	.84794651E 03	.13702051E-03	.11091211E 01	-.40645462E 00	.98385196E 03
Y	.12302116E 01	.84794652E 03	.14369910E-03	.11091211E 01	-.40645462E 00	.98385207E 03
S	.92095000E 03					
X	.12404019E 01	.87558621E 03	.47788507E-03	.11138091E 01	-.40304022E 00	.10218012E 04
Y	.12403937E 01	.87558564E 03	.46414403E-03	.11138091E 01	-.40304022E 00	.10218014E 04

S	.95167000E 03					
X	.12514426E 01	.90310842E 03	.35469437E-03	.11185775E 01	-.39961130E 00	.10600747E 04
Y	.12514382E 01	.90310807E 03	.37026275E-03	.11185775E 01	-.39961129E 00	.10600748E 04
S	.97983000E 03					
X	.12614042E 01	.92823351E 03	.36289819E-03	.11230212E 01	-.39645514E 00	.10954487E 04
Y	.12613998E 01	.92823316E 03	.37879785E-03	.11230212E 01	-.39645514E 00	.10954488E 04
S	.10079900E 04					
X	.12715646E 01	.95325861E 03	.39154333E-03	.11275363E 01	-.39328637E 00	.11311060E 04
Y	.12715583E 01	.95325823E 03	.40727932E-03	.11275363E 01	-.39328637E 00	.11311061E 04
S	.10399900E 04					
X	.12829083E 01	.98157375E 03	.35468101E-03	.11327564E 01	-.38966993E 00	.11719769E 04
Y	.12829139E 01	.98157347E 03	.33822195E-03	.11327564E 01	-.38966993E 00	.11719771E 04
S	.10719900E 04					
X	.12954404E 01	.10097576E 04	.41784745E-03	.11380745E 01	-.38603663E 00	.12132297E 04
Y	.12954333E 01	.10097572E 04	.43429377E-03	.11380745E 01	-.38603662E 00	.12132298E 04
S	.11039900E 04					
X	.13073653E 01	.10378086E 04	.31227597E-03	.11434940E 01	-.38238615E 00	.12548737E 04
Y	.13073765E 01	.10378085E 04	.29714951E-03	.11434940E 01	-.38238614E 00	.12548739E 04
S	.11359900E 04					
X	.13204120E 01	.10657260E 04	.53906425E-03	.11490181E 01	-.37871815E 00	.12969178E 04
Y	.13203952E 01	.10657255E 04	.55023891E-03	.11490181E 01	-.37871815E 00	.12969179E 04
S	.11679900E 04					
X	.13331434E 01	.10935078E 04	.21908348E-03	.11546508E 01	-.37503231E 00	.13393727E 04
Y	.13331648E 01	.10935079E 04	.21586898E-03	.11546508E 01	-.37503231E 00	.13393729E 04
S	.11987100E 04					
X	.13457478E 01	.11200503E 04	.45379382E-03	.11601633E 01	-.37147678E 00	.13805241E 04
Y	.13457474E 01	.11200500E 04	.43468038E-03	.11601633E 01	-.37147678E 00	.13805244E 04
S	.12332700E 04					
X	.13606979E 01	.11497584E 04	.23040361E-03	.11664929E 01	-.36745636E 00	.14272948E 04
Y	.13607206E 01	.11497585E 04	.23238275E-03	.11664929E 01	-.36745636E 00	.14272950E 04
S	.12639900E 04					
X	.13739214E 01	.11760294E 04	.36435051E-03	.11722368E 01	-.36386410E 00	.14693010E 04
Y	.13739330E 01	.11760293E 04	.34718698E-03	.11722369E 01	-.36386410E 00	.14693012E 04
S	.12947100E 04					
X	.13877148E 01	.12021709E 04	.56931341E-03	.11780956E 01	-.36025402E 00	.15117249E 04
Y	.13877036E 01	.12021704E 04	.55161851E-03	.11780956E 01	-.36025402E 00	.15117252E 04
S	.13292700E 04					
X	.14038152E 01	.12314229E 04	.26591152E-03	.11848292E 01	-.35617093E 00	.15599651E 04
Y	.14038393E 01	.12314230E 04	.26766693E-03	.11848292E 01	-.35617093E 00	.15599654E 04

S	.13638300E 04					
X	.14203716E 01	.12605078E 04	.61605302E-03	.11917192E 01	-.35206430E 00	.16087622E 04
Y	.14203533E 01	.12605073E 04	.63002444E-03	.11917192E 01	-.35206430E 00	.16087625E 04
S	.14009500E 04					
X	.14384329E 01	.12915578E 04	.67529865E-03	.11993016E 01	-.34762663E 00	.16618145E 04
Y	.14384094E 01	.12915573E 04	.68324718E-03	.11993016E 01	-.34762662E 00	.16618148E 04
S	.14342300E 04					
X	.14551118E 01	.13192270E 04	.31219982E-03	.12062668E 01	-.34362380E 00	.17099599E 04
Y	.14551371E 01	.13192272E 04	.31665648E-03	.12062668E 01	-.34362380E 00	.17099603E 04
S	.14713500E 04					
X	.14745411E 01	.13498990E 04	.40001043E-03	.12142300E 01	-.33913143E 00	.17643282E 04
Y	.14745567E 01	.13498991E 04	.41758808E-03	.12142300E 01	-.33913142E 00	.17643285E 04
S	.15123100E 04					
X	.14961436E 01	.13835081E 04	.49766566E-03	.12232664E 01	-.33413953E 00	.18251659E 04
Y	.14961517E 01	.13835081E 04	.47654032E-03	.12232664E 01	-.33413952E 00	.18251663E 04
S	.15532700E 04					
X	.15192773E 01	.14168663E 04	.76711732E-03	.12325778E 01	-.32911005E 00	.18869226E 04
Y	.15192498E 01	.14168658E 04	.76898211E-03	.12325778E 01	-.32911005E 00	.18869230E 04
S	.15942300E 04					
X	.15432504E 01	.14499692E 04	.54464081E-03	.12421796E 01	-.32404185E 00	.19496349E 04
Y	.15432564E 01	.14499691E 04	.56692606E-03	.12421796E 01	-.32404185E 00	.19496353E 04
S	.16351900E 04					
X	.15676992E 01	.14828132E 04	.41743134E-03	.12520878E 01	-.31893370E 00	.20133396E 04
Y	.15677280E 01	.14828135E 04	.41550348E-03	.12520878E 01	-.31893369E 00	.20133400E 04
S	.16761500E 04					
X	.15932114E 01	.15153945E 04	.58081699E-03	.12623196E 01	-.31378430E 00	.20780767E 04
Y	.15932204E 01	.15153946E 04	.55856556E-03	.12623197E 01	-.31378430E 00	.20780772E 04
S	.17171100E 04					
X	.16200502E 01	.15477084E 04	.77890193E-03	.12728944E 01	-.30859231E 00	.21438893E 04
Y	.16200333E 01	.15477082E 04	.75941874E-03	.12728944E 01	-.30859230E 00	.21438899E 04
S	.17580700E 04					
X	.16481384E 01	.15797504E 04	.88348498E-03	.12838327E 01	-.30335626E 00	.22108237E 04
Y	.16481095E 01	.15797502E 04	.87568332E-03	.12838327E 01	-.30335625E 00	.22108243E 04
S	.17990300E 04					
X	.16774308E 01	.16115159E 04	.92672106E-03	.12951569E 01	-.29807462E 00	.22789288E 04
Y	.16773997E 01	.16115156E 04	.92735896E-03	.12951569E 01	-.29807461E 00	.22789294E 04
S	.18399900E 04					
X	.17079828E 01	.16429998E 04	.95965633E-03	.13068916E 01	-.29274577E 00	.23482570E 04
Y	.17079513E 01	.16429996E 04	.96218333E-03	.13068916E 01	-.29274576E 00	.23482576E 04

S	.18809500E 04					
X	.17398892E 01	.16741972E 04	.99379491E-03	.13190635E 01	-.28736796E 00	.24188648E 04
Y	.17398572E 01	.16741970E 04	.99124596E-03	.13190635E 01	-.28736796E 00	.24188655E 04
S	.19219100E 04					
X	.17732685E 01	.17051026E 04	.99332661E-03	.13317019E 01	-.28193936E 00	.24908130E 04
Y	.17732417E 01	.17051026E 04	.97934034E-03	.13317019E 01	-.28193935E 00	.24908138E 04
S	.19628700E 04					
X	.18083281E 01	.17357104E 04	.89997874E-03	.13448391E 01	-.27645798E 00	.25641671E 04
Y	.18083227E 01	.17357107E 04	.87563786E-03	.13448391E 01	-.27645797E 00	.25641680E 04
S	.20038300E 04					
X	.18454073E 01	.17660148E 04	.75899095E-03	.13585106E 01	-.27092171E 00	.26389979E 04
Y	.18454345E 01	.17660153E 04	.74435543E-03	.13585106E 01	-.27092170E 00	.26389988E 04
S	.20447900E 04					
X	.18846603E 01	.17960096E 04	.84646027E-03	.13727554E 01	-.26532827E 00	.27153815E 04
Y	.18846845E 01	.17960102E 04	.86402210E-03	.13727554E 01	-.26532827E 00	.27153824E 04
S	.20857500E 04					
X	.19256215E 01	.18256884E 04	.11967011E-02	.13876168E 01	-.25967525E 00	.27934008E 04
Y	.19255935E 01	.18256886E 04	.12114041E-02	.13876168E 01	-.25967524E 00	.27934018E 04
S	.21267100E 04					
X	.19685367E 01	.18550438E 04	.10989810E-02	.14031439E 01	-.25396001E 00	.28731480E 04
Y	.19685316E 01	.18550444E 04	.10741740E-02	.14031439E 01	-.25396000E 00	.28731492E 04
S	.21676700E 04					
X	.20148586E 01	.18840689E 04	.10104442E-02	.14193900E 01	-.24817971E 00	.29547209E 04
Y	.20148867E 01	.18840698E 04	.10260500E-02	.14193900E 01	-.24817970E 00	.29547221E 04
S	.22086300E 04					
X	.20631918E 01	.19127563E 04	.14070941E-02	.14364154E 01	-.24233129E 00	.30382269E 04
Y	.20631568E 01	.19127569E 04	.14008428E-02	.14364154E 01	-.24233129E 00	.30382282E 04
S	.22495900E 04					
X	.21150329E 01	.19410970E 04	.11129760E-02	.14542887E 01	-.23641143E 00	.31237871E 04
Y	.21150691E 01	.19410982E 04	.11174973E-02	.14542887E 01	-.23641143E 00	.31237886E 04
S	.22905500E 04					
X	.21698240E 01	.19690833E 04	.15519419E-02	.14730856E 01	-.23041650E 00	.32115298E 04
Y	.21697907E 01	.19690842E 04	.15406444E-02	.14730856E 01	-.23041649E 00	.32115314E 04
S	.23315100E 04					
X	.22290049E 01	.19967051E 04	.14164965E-02	.14928941E 01	-.22434253E 00	.33016024E 04
Y	.22290217E 01	.19967065E 04	.14392278E-02	.14928941E 01	-.22434252E 00	.33016042E 04
S	.23750300E 04					
X	.22955394E 01	.20256436E 04	.17653052E-02	.15151592E 01	-.21779748E 00	.34000357E 04
Y	.22955139E 01	.20256450E 04	.17518709E-02	.15151593E 01	-.21779747E 00	.34000377E 04

S	.24159900E 04					
X	.23638218E 01	.20524825E 04	.18036635E-02	.15373826E 01	-.21154631E 00	.34954404E 04
Y	.23638050E 01	.20524844E 04	.18222473E-02	.15373826E 01	-.21154630E 00	.34954426E 04
S	.24569500E 04					
X	.24368266E 01	.20789248E 04	.17004763E-02	.15609702E 01	-.20520137E 00	.35937294E 04
Y	.24368504E 01	.20789273E 04	.17158507E-02	.15609702E 01	-.20520136E 00	.35937319E 04
S	.25030300E 04					
X	.25260476E 01	.21081834E 04	.22285979E-02	.15893281E 01	-.19794386E 00	.37080356E 04
Y	.25260153E 01	.21081859E 04	.22251800E-02	.15893281E 01	-.19794386E 00	.37080384E 04
S	.25439900E 04					
X	.26128181E 01	.21337411E 04	.23665912E-02	.16163564E 01	-.19137924E 00	.38132485E 04
Y	.26127860E 01	.21337442E 04	.23719372E-02	.16163564E 01	-.19137923E 00	.38132517E 04
S	.25849500E 04					
X	.27072952E 01	.21588603E 04	.25732585E-02	.16453224E 01	-.18470001E 00	.39221670E 04
Y	.27072624E 01	.21588640E 04	.25783856E-02	.16453224E 01	-.18470001E 00	.39221707E 04
S	.26259100E 04					
X	.28106621E 01	.21835253E 04	.28517124E-02	.16764830E 01	-.17789774E 00	.40351356E 04
Y	.28106285E 01	.21835298E 04	.28473374E-02	.16764830E 01	-.17789774E 00	.40351398E 04
S	.26719900E 04					
X	.29399480E 01	.22107085E 04	.28294851E-02	.17145493E 01	-.17008611E 00	.41675669E 04
Y	.29399538E 01	.22107144E 04	.28511807E-02	.17145493E 01	-.17008610E 00	.41675718E 04
S	.27206300E 04					
X	.30929148E 01	.22387240E 04	.34547272E-02	.17587684E 01	-.16164081E 00	.43142078E 04
Y	.30929238E 01	.22387313E 04	.34317142E-02	.17587684E 01	-.16164080E 00	.43142137E 04
S	.27743900E 04					
X	.32882872E 01	.22688348E 04	.37563229E-02	.18134380E 01	-.15204170E 00	.44856174E 04
Y	.32883353E 01	.22688447E 04	.37574987E-02	.18134380E 01	-.15204169E 00	.44856247E 04
S	.28383900E 04					
X	.35665187E 01	.23034358E 04	.49645927E-02	.18884533E 01	-.14020210E 00	.47046691E 04
Y	.35664618E 01	.23034495E 04	.49634992E-02	.18884533E 01	-.14020209E 00	.47046788E 04
S	.29151900E 04					
X	.39907101E 01	.23430136E 04	.61657076E-02	.19976333E 01	-.12529517E 00	.49940864E 04
Y	.39907276E 01	.23430363E 04	.61994884E-02	.19976332E 01	-.12529516E 00	.49941007E 04
S	.29971100E 04					
X	.46143089E 01	.23826202E 04	.91137849E-02	.21482247E 01	-.10834348E 00	.53450114E 04
Y	.46143727E 01	.23826619E 04	.90945560E-02	.21482248E 01	-.10834347E 00	.53450356E 04
S	.30790300E 04					
X	.55587337E 01	.24191128E 04	.14348888E-01	.23577667E 01	-.89939868E-01	.57587318E 04
Y	.55588075E 01	.24192034E 04	.14374463E-01	.23577666E 01	-.89939857E-01	.57587779E 04

S	.31609500E 04					
X	.72021043E 01	.24518395E 04	.27819327E-01	.26839296E 01	-.69403453E-01	.62742056E 04
Y	.72021733E 01	.24521005E 04	.27824056E-01	.26839296E 01	-.69403441E-01	.62743203E 04
S	.32428700E 04					
X	.11060675E 02	.24796293E 04	.81244790E-01	.33266381E 01	-.45154328E-01	.69943467E 04
Y	.11060341E 02	.24811135E 04	.81730564E-01	.33266287E 01	-.45154308E-01	.69948590E 04
S	.33299100E 04					
X	.59416363E 02	.25001005E 04	.39364292E 01	.81138123E 01	-.53116288E-02	.88378975E 04
Y	-.63377109E 03	.11813890E 05	.19131357E 02	-.85517932E 01	-.50895876E-02	.89911212E 04
S	-.21062458E 66					
X	-.21062458E 66	-.21062458E 66	-.57896045E 77	-.21062458E 66	-.57896045E 77	-.21062458E 66
Y	-.57896045E 69	-.57896045E 77	-.20000000E-03	-.31863985E 13	-.11579209E 69	-.39829981E 25
S	-.21062458E 66					
X	-.21062458E 66	-.21062458E 66	-.57896045E 77	-.21062458E 66	-.57896045E 77	-.21062458E 66
Y	-.57896045E 69	-.57896045E 77	-.20000000E-03	-.31863985E 13	-.11579209E 69	-.39829981E 25

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APPROVAL

ASYMPTOTIC SOLUTION TO THE TANGENTIAL LOW THRUST ENERGY INCREASE TRAJECTORY

By Klaus J. Schwenzfeger

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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